Loans & Annuities

Notation and Terminology

L : amount of a loan in dollars

x : monthly payment or deposit

r : yearly interest rate as a decimal

n : total number of payments or deposits

Loan Matrix

Functional Form:

$$T\begin{bmatrix} L \\ x \end{bmatrix} = \begin{bmatrix} L + L\left(\frac{r}{12}\right) - x \\ x \end{bmatrix}$$

Matrix Form:

$$T = \left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \boxed{\begin{bmatrix} 1+\frac{r}{12} & -1 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1+\frac{r}{12} & -1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} L \\ x \end{pmatrix} = \begin{pmatrix} L + L\left(\frac{r}{12}\right) - x \\ x \end{pmatrix} = \begin{pmatrix} \textit{balance after} \\ \textit{one payment} \\ x \end{pmatrix}$$

$$\begin{bmatrix} 1+\frac{r}{12} & -1 \\ 0 & 1 \end{bmatrix}^2\begin{pmatrix} L \\ x \end{pmatrix} = \begin{bmatrix} 1+\frac{r}{12} & -1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} L + L\left(\frac{r}{12}\right) - x \\ x \end{pmatrix}$$

$$= \begin{pmatrix} L + L\frac{r}{12} - x + \left[L + L\frac{r}{12} - x\right]\frac{r}{12} - x \\ x \end{pmatrix} = \begin{pmatrix} \textit{balance after} \\ \textit{two payments} \\ x \end{pmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 1+\frac{r}{12} & -1 \\ 0 & 1 \end{bmatrix}^n\begin{pmatrix} L \\ x \end{pmatrix} = \begin{pmatrix} f(L, r, n, x) \\ x \end{pmatrix} = \begin{pmatrix} \textit{balance after} \\ n \textit{ payments} \\ x \end{pmatrix}$$

Example 1

Suppose you have just bought a house for \$420,000. After putting a down payment of \$30,000 you take out a mortgage for the remaining \$390,000 at an interest rate of 4.7% per year. Use matrices to determine the monthly payment that will pay this loan in

a. 30 years (360 monthly installments).

$$\begin{bmatrix} 1+\dfrac{0.047}{12} & -1 \\ 0 & 1 \end{bmatrix}^{360} \begin{bmatrix} 390000 \\ x \end{bmatrix} = \begin{bmatrix} f(x) \simeq 1.593 \cdot 10^6 - 787.58x \\ x \end{bmatrix}$$

For the loan to be paid off, the outstanding balance must be 0. Therefore,

$$1.59302945096 \; - \; 787.58x = \; 0 \quad \Leftrightarrow \quad x = \frac{1.59302945096}{787.58} \simeq \; 2,022.69$$

The monthly payment to pay off \$390,000 in 30 years is \$2,022.69 ($r = 4.7\%$).

b. 20 years (240 monthly installments).

$$\begin{bmatrix} 1+\dfrac{0.047}{12} & -1 \\ 0 & 1 \end{bmatrix}^{240} \begin{bmatrix} 390000 \\ x \end{bmatrix} = \begin{bmatrix} f(x) \simeq 9.9656 \cdot 10^5 - 397.09x \\ x \end{bmatrix}$$

$$x = \frac{996561.424529131}{397.0941884824} \simeq \; 2,509.63$$

The monthly payment to pay off \$390,000 in 20 years is \$2,509.63.

c. 5 years (60 monthly installments).

$$\begin{bmatrix} 1+\dfrac{0.047}{12} & -1 \\ 0 & 1 \end{bmatrix}^{60} \begin{bmatrix} 390000 \\ x \end{bmatrix} = \begin{bmatrix} f(x) \simeq 493088.04524 - 67.48808x \\ x \end{bmatrix}$$

$$x = \frac{493088.04524}{67.48808} \simeq \; 7,306.30$$

The monthly payment to pay off \$390,000 in 5 years is \$7,306.30.

Example 2

With reference to **Example 1**, determine the amount outstanding on loan (a) 15 years into the loan (180 payments have been made).

$$\begin{bmatrix} 1+\dfrac{0.047}{12} & -1 \\ 0 & 1 \end{bmatrix}^{180} \begin{bmatrix} 390000 \\ 2022.69 \end{bmatrix} = \begin{bmatrix} 260905.68 \\ 2022.69 \end{bmatrix}$$

Therefore, the outstanding balance of the loan after 15 years will be $ 260,905.68.

Example 3

With reference to **Example 1**, determine the amount outstanding on loan (b) 5 years into the loan (60 payments have been made).

$$\begin{bmatrix} 1+\dfrac{0.047}{12} & -1 \\ 0 & 1 \end{bmatrix}^{60} \begin{bmatrix} 390000 \\ 2509.63 \end{bmatrix} = \begin{bmatrix} 323717.94 \\ 2509.63 \end{bmatrix}$$

Therefore, the outstanding balance of the loan after 5 years will be $ 323,717.94.

Example 4

With reference to **Example 1**, determine the amount outstanding on loan (c) after 50 payments have been made.

$$\begin{bmatrix} 1+\dfrac{0.047}{12} & -1 \\ 0 & 1 \end{bmatrix}^{50} \begin{bmatrix} 390000 \\ 7306.30 \end{bmatrix} = \begin{bmatrix} 71513.37 \\ 7306.30 \end{bmatrix}$$

Therefore, the outstanding balance of the loan after 50 payments will be $ 71,513.37.

Annuities

An **ordinary annuity** is a sequence of periodic deposits (usually monthly) made at the **end** of each period (month), for the purpose accumulating what we call a **future value** to be used either for retirement, to finance college education, or some other purpose.

Annuity Matrix

$$T = \begin{bmatrix} 1+\frac{r}{12} & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+\frac{r}{12} & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} \textit{future value} \\ \textit{after one deposit} \\ x \end{pmatrix}$$

$$\begin{bmatrix} 1+\frac{r}{12} & 1 \\ 0 & 1 \end{bmatrix}^2 \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{bmatrix} 1+\frac{r}{12} & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} x + x\frac{r}{12} + x \\ x \end{pmatrix} = \begin{pmatrix} \textit{future value} \\ \textit{after two deposits} \\ x \end{pmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 1+\frac{r}{12} & 1 \\ 0 & 1 \end{bmatrix}^n \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} f(r, n, x) \\ x \end{pmatrix} = \begin{pmatrix} \textit{future value} \\ \textit{after } n \textit{ deposits} \\ x \end{pmatrix}$$

Example 1

Find the future value of an ordinary annuity involving deposits of \$300 per month at a yearly interest rate of 5.5% per year after

a. ten years.

$$\begin{bmatrix} 1+\frac{0.055}{12} & 1 \\ 0 & 1 \end{bmatrix}^{120} \begin{bmatrix} 0 \\ 300 \end{bmatrix} = \begin{bmatrix} 47852.27 \\ 300 \end{bmatrix}$$

The future value of the annuity after 10 years will be \$47,852.27.

b. twenty-five years.

$$\begin{bmatrix} 1+\frac{0.055}{12} & 1 \\ 0 & 1 \end{bmatrix}^{300} \begin{bmatrix} 0 \\ 300 \end{bmatrix} = \begin{bmatrix} 192611.22 \\ 300 \end{bmatrix}$$

The future value of the annuity after 25 years will be $192,611.22.

Example 2

Starting today, you decide to deposit $450 per month in an ordinary annuity that earns annual interest of 6.5%. After 40 years you retire and your contributions stop. However, you remain invested at the more conservative rate of 4.5%.

a. How much would the annuity be worth at the time you retire?

$$\begin{bmatrix} 1+\frac{0.065}{12} & 1 \\ 0 & 1 \end{bmatrix}^{480} \begin{bmatrix} 0 \\ 450 \end{bmatrix} = \begin{bmatrix} 1027628.63 \\ 450 \end{bmatrix}$$

At retirement, the future value of this annuity will be $1,027,628.63.

b. Suppose you receive a pension of $5,000 a month in retirement for a period of 30 years. Would you have run out of funds by then? Explain.

$$\begin{bmatrix} 1+\frac{0.045}{12} & -1 \\ 0 & 1 \end{bmatrix}^{360} \begin{bmatrix} 1027628.63 \\ 5000 \end{bmatrix} = \begin{bmatrix} 157073.94 \\ 5000 \end{bmatrix}$$

No, at that point the annuity is worth $157,073.94.

c. What is the maximum pension you can receive for exactly 30 years?

$$\begin{bmatrix} 1+\frac{0.045}{12} & -1 \\ 0 & 1 \end{bmatrix}^{360} \begin{bmatrix} 1027628.63 \\ x \end{bmatrix} = \begin{bmatrix} 3954004.67574-759.386x \\ x \end{bmatrix}$$

$$3954004.67574-759.386x = 0 \quad \Leftrightarrow \quad x \simeq 5206.84$$

Maximum pension: $5,206.84 (per month for 30 years)